Real Analysis II
Problem Set 1: Topology I

Exam style questions are marked with (*).

Open, Closed, and Compact sets

Consider the following two definitions of limit points.

Definition. A point $x \in \mathbb{R}$ is a limit point of a subset $A \subseteq \mathbb{R}$ if $\lim_{n\to\infty} a_n = x$ for some sequence (a_n) where $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \neq x$ for all $n \in \mathbb{N}$.

Definition. A point $x \in \mathbb{R}$ is a limit point of a subset $A \subseteq \mathbb{R}$ if every ε-neighbourhood $V_\varepsilon(x)$ has an intersection with A containing some point other than x, i.e.

$$\big(A \cap V_\varepsilon(x)\big) \setminus \{x\} \neq \emptyset.$$

Question 1. Prove the two preceding definitions are equivalent.

Question 2. The rationals are *dense* in $\mathbb{R}$, meaning in any interval of real numbers (a, b), with $a < b$, there is some rational number in the interval. Prove that this is equivalent to the following definition: for every $x \in \mathbb{R}$ there exists a Cauchy sequence of rationals converging to x.

Question 3. Prove the *Nested Interval property* holds for nested compact sets. That is, that for any collection of non-empty, nested, compact sets K_n with $K_n \subseteq K_{n-1}$ for all $n \in \mathbb{N}$, the intersection is non empty, i.e.

$$\cap K_n \neq \emptyset.$$

Metric Spaces (Heine-Borel)

Question 4. Consider $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{Q}$. We are no longer considering these as subsets of $\mathbb{R}$, but spaces in their own right. The notion of open and closed for such spaces rely on a definition of an appropriate distance function, called a *metric*. Here we will just use standard distance, and so we carry over our definitions of ε-neighbourhoods.

- We call a subset S of a metric space M *open* if for every $s \in S$ there exists some $\varepsilon > 0$ such that $V_\varepsilon(s) \subset S$.
- We call S *closed* if the complement of S in M is open.

Consider the subsets $S_1 = (0, 1] \subset \mathbb{R}^*$ and $S_2 = [0, 1] \subset \mathbb{Q}$. Show both of these subsets are closed and bounded in their respective spaces. Decide whether every open cover of S_1 and S_2 in their respective spaces have finite subcovers. Decide whether the Heine-Borel theorem applies in these spaces.

Limits

Question 5. Describe with the help of a graph the ε-δ definition of a limit of a real valued function.

Question 6. Find the following limits:

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{\cdots}}}}; \qquad \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1}.$$